NON-LINEAR SYSTEMS

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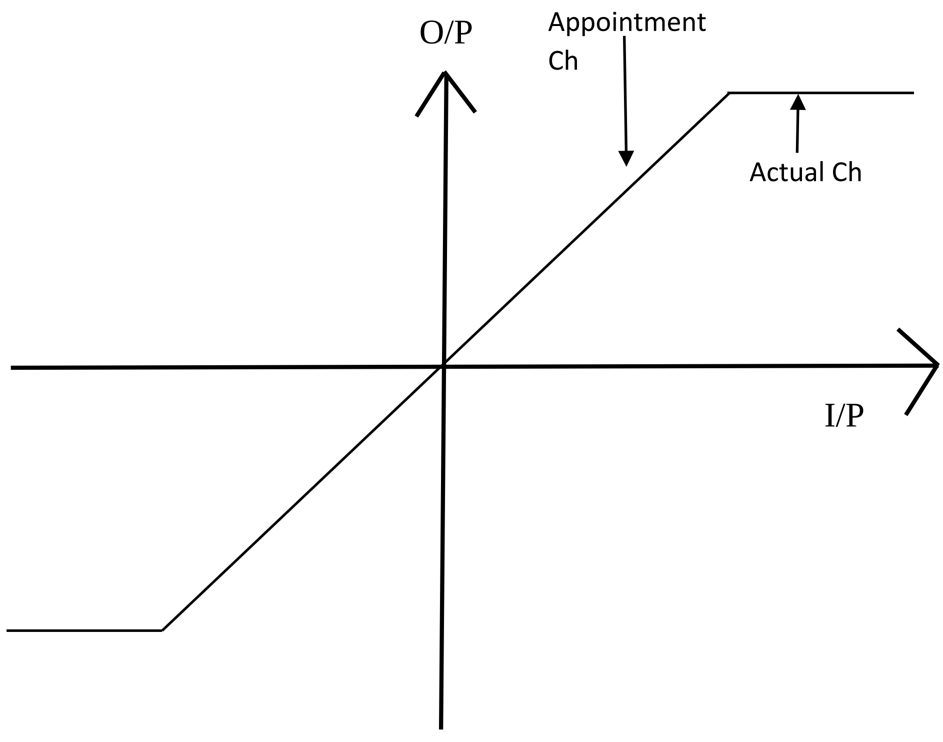
MODULE 1

Classification of Non-Linearities

Non-linearities can be classified into two types

1. **Incidental Non-Linearities:** These are the non-linearities which are inherently present in the system.
   Ex: Saturation Dead Zone, Coulomb Friction, Stiction and Backlash.
2. **Intentional Non-Linearities:** These are the non-linearities which are deliberately inserted in the system to modify the system characteristics.
   Ex: Relay

Saturation



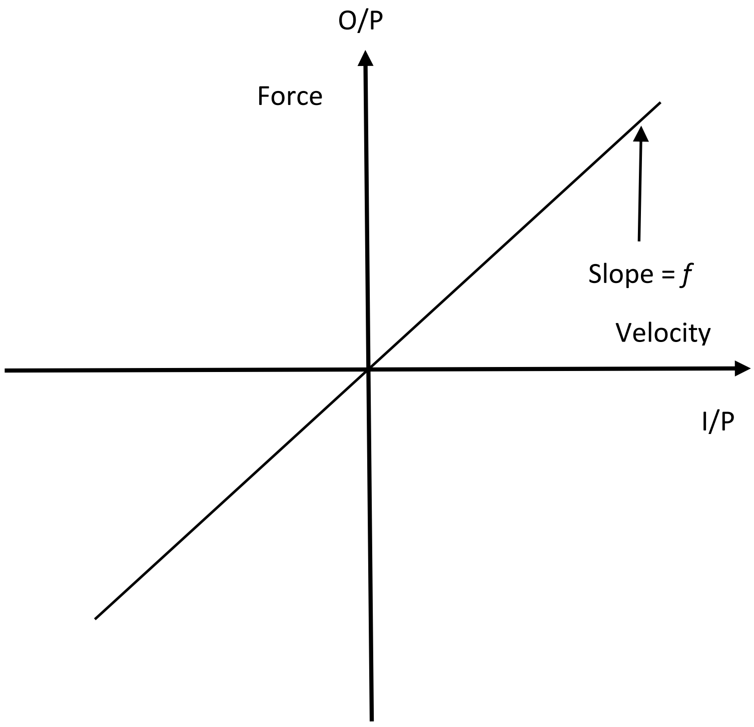
* Output is proportional to the input in the limited range of input signals.
* When input exceeds the range, the output tends to become nearly constants.

Friction

$$Viscous\ Friction\ Force = f_{xi}$$

Where f is constant and x is relative velocity.

Viscous Friction is then linear in nature



- ∗ Addition to the viscous force, there exists two non-linear friction
    a. Coulomb Friction
    b. Stiction
- ➢ Coulomb Friction which is a constant retarding force (always opposing the relative motion).
- ➢ Stiction Friction which is the force required to initiate motion.

$$Stiction\ Force\ > Coulomb\ Force$$

- ∗ In actual practice, stiction force decreases with velocity and changes over to Coulomb force at reasonably low force.

In actual practice, Frictional Force will proportional to square and cube of the speed.

Relay (Intentional Non-Linearity)

- → A relay is non-linear power amplifier which can provide large power amplification inexpensively and is deliberately introduced in control system.
- → A relay controlled system can be switched abruptly between several discrete states.
- → Relay-controlled system has wide applications in the control field.

Jump Phenomenon:

Non-linear system exhibit phenomenon that cannot exist in linear system. The amplitude of variation can increase or decrease abruptly as the excitation frequency ω is increased or decreased. This is known as jump phenomenon.

Singular Point:

Time variant system is described by the state equation

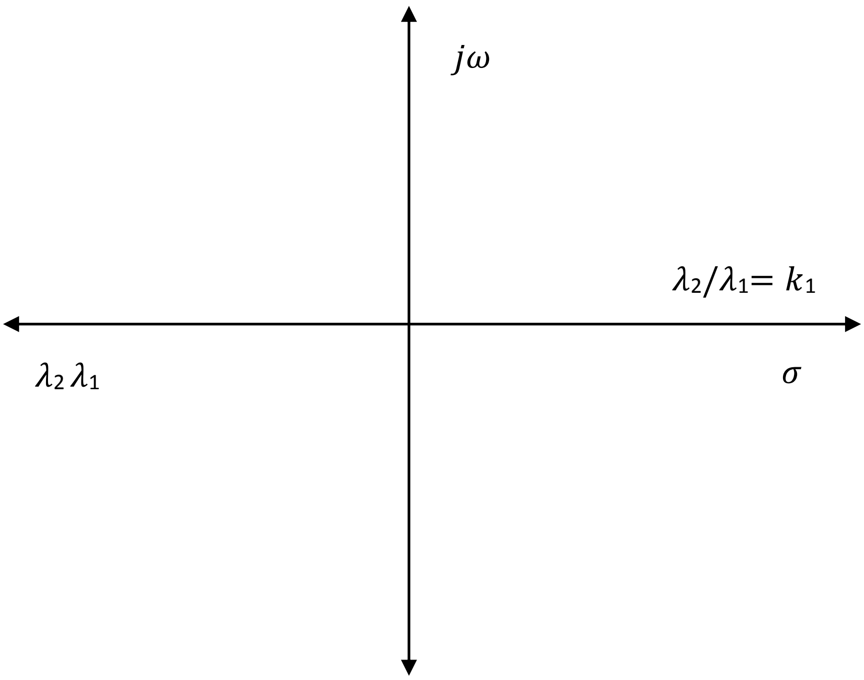
$$x = f(x, u)$$

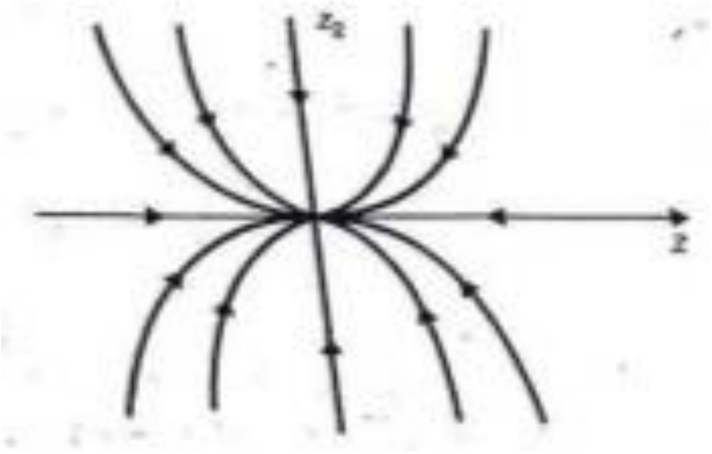
$$x = f(x)$$

* A system represented by an equation of the above form called as autonomous system.
* The points in the phase-space at which the derivatives of all the state variables are zero. Such points are called singular points.

Nodal Points:

Eigen Values (λ_1, λ_2) are both real and negative.





* They all converge to origin which is then called as nodal point. This is stable nodal point.
* When λ_1 & λ_2 are both real and positive the result is an unstable nodal point.

Saddle Point

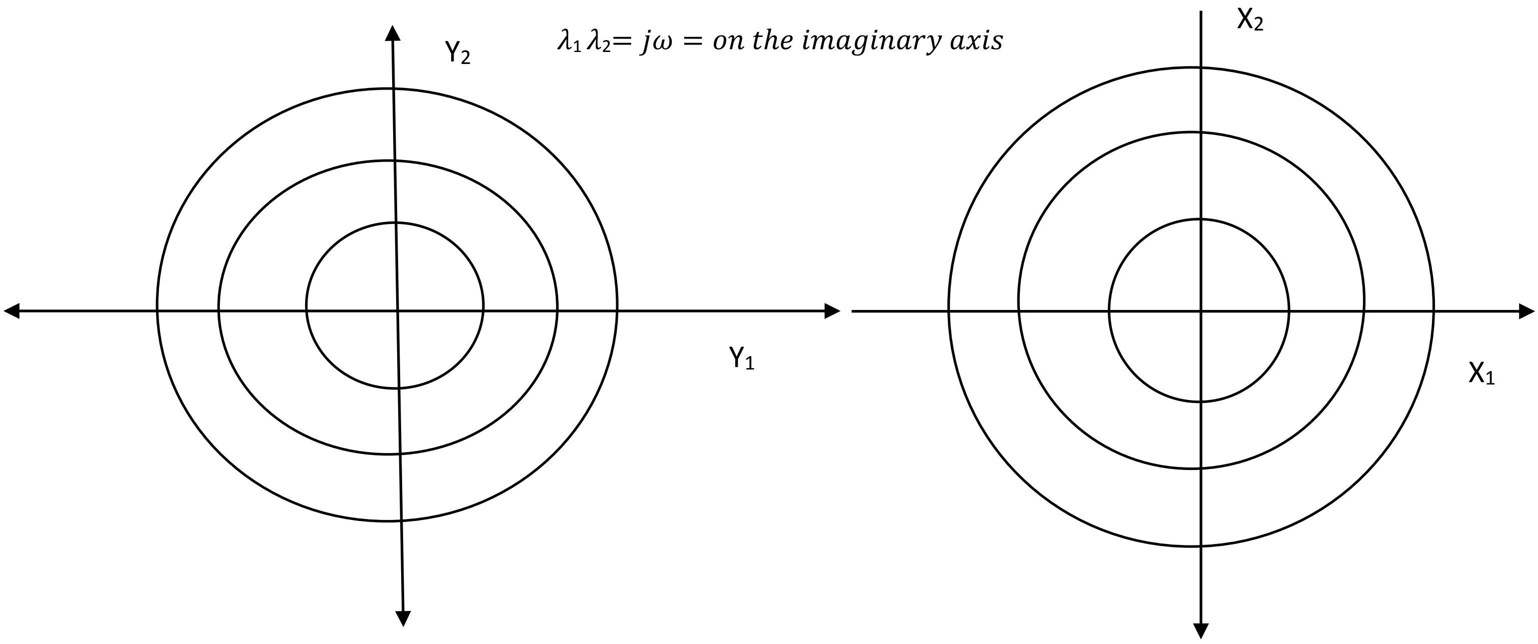
Both eigen values are real, equal and negative of each other. The origin in this case a saddle point which is always unstable, one eigenvalue being negative.

Focus Point:

$\lambda_1, \lambda_2 = \sigma \pm j\omega = complex\ conjugate\ pair$

* The origin in the focus point and is stable/unstable for negative/positive real parts of the eigen values.
* Transformation has been carried out for $(x_1 x_2)$ to $(y_1\ y_2)$ to present the trajectory inform of spiral.

Centre/Vertex Point:



Stability of Non-Linear System:

i. For Free System: A system is stable with zero input and arbitrary initial conditions if the resulting trajectory tends to the equilibrium state.

ii. Force System: A system is stable if with bounded input, the system output is bounded.

There are types of stability condition in literature. Basically we concentrated on the following:

i. Stability, Asymptotic and Asymptotic in the large.

$x = f(x) \longrightarrow$ Asymptotic in the large

Because $x(t_0) = 0$

$X(t)$ remain near the origin for all $'t'$

ii A system is said to be locally stable (stable in the small) if the resign $\delta(t)$ is small.

→ It is stable

→ Every initial state $X(t_0)$ results in $X(t) \to 0$ as $t \to \infty$

* Asymptotically Stability in the large guarantees that every motion will approach the origin.

Limit Cycles:

System Stability has been defined in terms of distributed steady- state coming back to its equilibrium position or at least staying within tolerable limits from it**.**

→ Distributed NL system while staying within tolerable limits may exhibit a special behaviour of closed trajectory/limit cycle.

The limit cycles describe the oscillations of non-linear system. The existence of a limit cycle corresponds to an oscillation of fixed amplitude and period.

Ex: Let us consider the well known Vander Pol's differential equation

$$\frac{d^2x}{d^2y} - \mu(1-x^2)\frac{d_x}{d_t} + x = 0 \qquad (1)$$

It describes the many non-linearities by comparing with the following linear differential equation:

$$\frac{d^2x}{dt^2} + 2\zeta\frac{d_x}{d_t} + x = 0$$

$$2\xi = -\mu(1-x^2) \Rightarrow \xi = -\left(\frac{\mu}{2}\right)(1-x^2) \qquad \text{\{Damping Factor\}}$$

a. $|x| \gg 1$, then damping factor has large positive means system behaves overdamped system with decrease of the amplitude of x_1. In this process the damping factor also decreases and system state finally reaches the limit cycle.

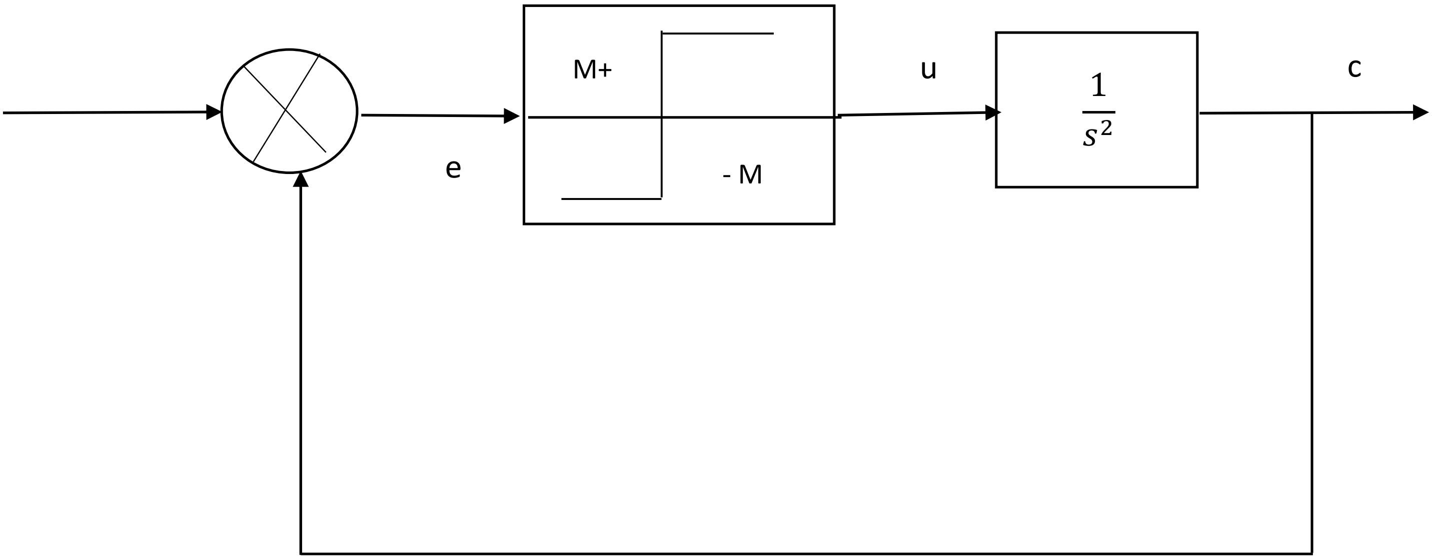
b. $|x| \ll 1$, then damping factor is negative, hence the amplitude of x increases till the system state again enters the limit cycle.

   ➢ On the other hand, if the paths in the neighbourhood of a limit cycle diverse away from it, it indicates that the limit cycle is unstable.
   Ex: Vander Pol Equation

$$\frac{d^2x}{d^2y} + \mu(1-x^2)\frac{d_x}{d_t} + x = 0$$

* It is important to note that a limit cycle in general is an undesirable characteristics of a control system. It may be tolerated only if its amplitude is within specified limits.

❖ Limit cycles of fixed amplitude and period can be sustained over a finite range of system parameter.



The differential equation describes the dynamics of the system which is given by

$$c = u$$

$$r - c = e$$

$$-\dot{c} = \dot{e} \Longrightarrow -\ddot{c} = \ddot{e} \text{ (as } r \text{ constant)}$$

Therefore $e = -u$

Choosing the state vector $[x_1 \, x_2]^T = [e \, \dot{e}]^T$, then we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{e} = -u$$

The ouput of on/off controller is

$$u = \mu sign(e) = \mu sign(x_1)$$

The state equation is described as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\mu sgn(x_1)$$

The slope of the equation is given by

$$\frac{dx_2}{dx_1} = \frac{\mu sign(x_1)}{x_2}$$

Separating the variables and integrating, we get

$$\int_{x_2(0)}^{x_2} x_2 dx_{\ 2} = -\mu \int_{x_1(0)}^{x_1} sign(x_1) dx$$

Carrying out integration and rearranging we get

$$x_2^2 = 2\mu x_1(0) + x_2^2(0);\ x_1 < 0$$

Region $I(x_1 = e\ positive, relay\ ouput + \mu)$

$$x_2^2 = 2\mu x_1 - 2\mu x_1(0) + x_2^2(0); x < 0$$

Region II $(x_1 = e\ neagtive, relay\ output - \mu)$

Construction by Graphical Methods:

1. **Isocline Method**
2. **$\boldsymbol{\delta}$ − Method**

1. **Isocline Method:**

→ Trajectory slope at any point is given by

$\frac{dx_2}{dx_1} = s = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$ (1)

→ Slope at a specific point is expressed as

$f_2(x_1, x_2) = S_1 f_1(x_1, x_2)$ (2)

* Given S_1, this is the equation of an isocline where any trajectory crosses at slope S_1.
* If isoclines for different values of S are drawn throughout the phase plane trajectory starting at any point can be constructed by drawing short lines from one isocline to another at average slope of the two adjoining isoclines.

2. **$\boldsymbol{\delta - Method}$:**

→ This method is used to construct a single trajectory for systems with describing differential equation in the form

$$\ddot{x} + f(x, \dot{x}, t) = 0$$

Converting this equation to the form

$$\ddot{x} + k^2[x + \delta(x, \dot{x}, t)] = 0$$

Let $k = W_n$, undamped natural frequency of the system when $\delta = 0$

Then the equation is written as

$$\ddot{x} + \omega_n^2[x + \delta(x, \dot{x}, t)] = 0$$

$$\ddot{x} + \omega_n^2(x + \delta) = 0$$

Choose the state variables as

$$\dot{x_1} = x, x_2 = \frac{\dot{x}}{\omega_n}$$

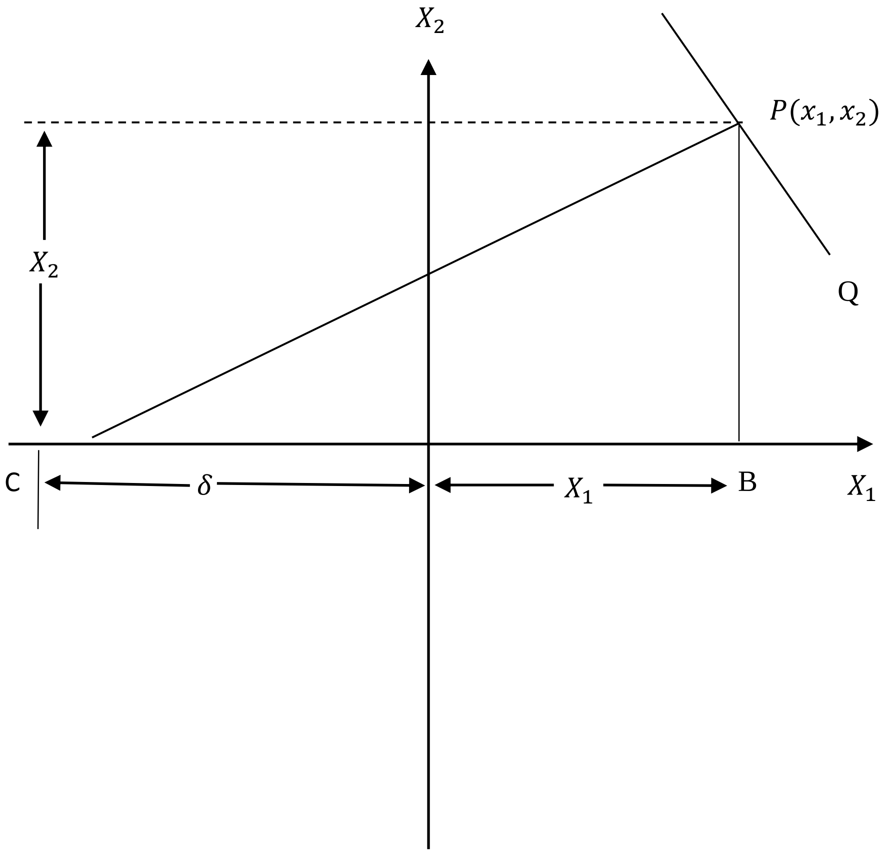
Giving state equation

$\dot{x_1} = \omega_n x_2$

$x_2 = -\omega_n(x_1 + \delta)$

The trajectory slope is expressed as

$\frac{dx_2}{dx_1} = -\frac{x_1 + \delta}{x_2}$



Any point $P(x_1, x_2)$, δ is computed and marked on negative side of $x_1 - axis$

→ The line of slope $-\frac{x_2}{(x_1+\delta)}$ is at 90^0 to CP. The trajectory at P is then identified as a short arc at P with centre at ‘C’.

→ The process is repeated for another ‘P’. shortly away from P(original) on the arc. Then completed trajectory can be drawn.

Phase Plane Method

Unforced linear spring mass damp system whose dynamics is described by

$$\mu\frac{d^2x}{dt^2} = 2\xi\omega n\frac{dx}{dt} + \omega_n^2 x = 0$$

ξ is the damping factor and ω_n =undamped natural frequency.

Let the state of the system is described by two variables, displacement (x) and the velocity$\left(\frac{dx}{dt}\right)$ in the state variable notation $x_1 = x, x_2 = \frac{dx}{dt}$,

$\Longrightarrow state\ variables = phase\ variables$

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_1}{dt} = -\omega_n^2 x_1 - 2\xi\omega_n x_2$$

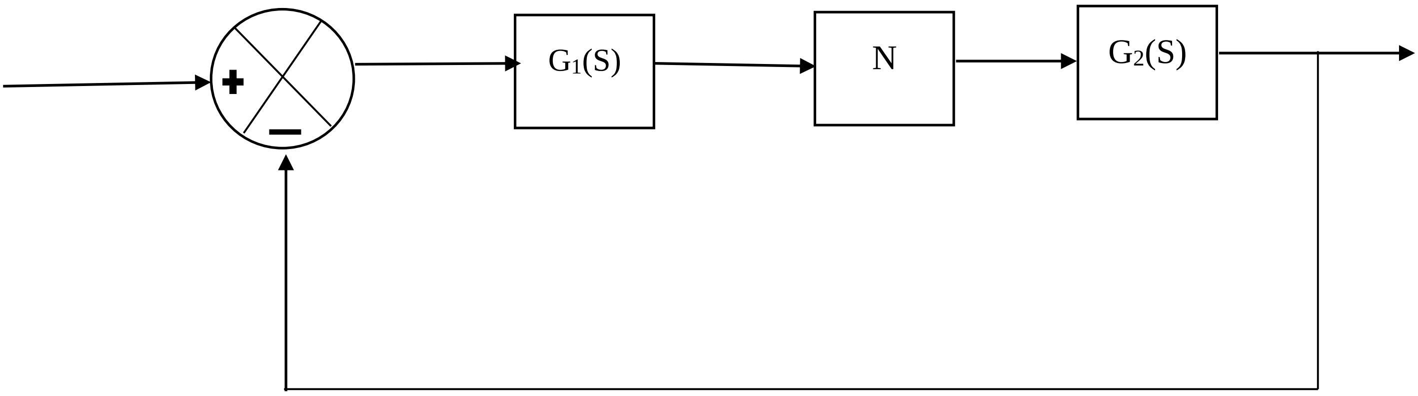
$$\frac{dx_1}{dt} = x_2$$

$\frac{dx_2}{dt} = -\frac{f}{\mu}x_2 - \frac{k_1x_1}{\mu} - \frac{k_2x_1^3}{\mu}$ (if the spring force is NL then $k_1x + k_2x^3$)

- ∗ The coordinate plane with axes that corresponds to the dependent variables $x_1 = x$ and its first derivative $x_2 = \ddot{x}$ is called phase-plane.
- ∗ The curve described by the state point (x_1, x_2) in the phase plane with time running parameter is called phase-trajectory.
- → A plane trajectory can be easily constructed by graphical or analytical technique.
- → Family of trajectories is called phase-portrait.

The phase plane method uses state model of the system which is obtained from the differential equation governing the system dynamics.

- → This method is powerful generally restricted to systems described by 2nd order differential equation.



→ $x = Xsin\omega t$

→ input x to the non-linearity is sinusoidal

→ Input, output of the non-linear element will in general be non-sinusoidal periodic function which maybe expressed in terms of Fourier series.

$$y = A_0 + A_1 sin\omega t + B_1 cos\omega t + A_2 sin2\omega t + B_2 cos2\omega t$$

→ If non-Linearity is assumed to be symmetrical. The average value of Y is zero, so that the outputs y is given by

$$y = A_1 sin\omega t + B_1 cos\omega t + A_2 sin2\omega t + B_2 cos2\omega t + \cdots$$

→ The harmonic of y can be thrown away for the purpose of analysis and the fundamental components of y i.e.

$$y_1 = A_1 sin\omega t + B_1 cos\omega t$$
$$= Y_1 \sin(\omega t + \phi)$$

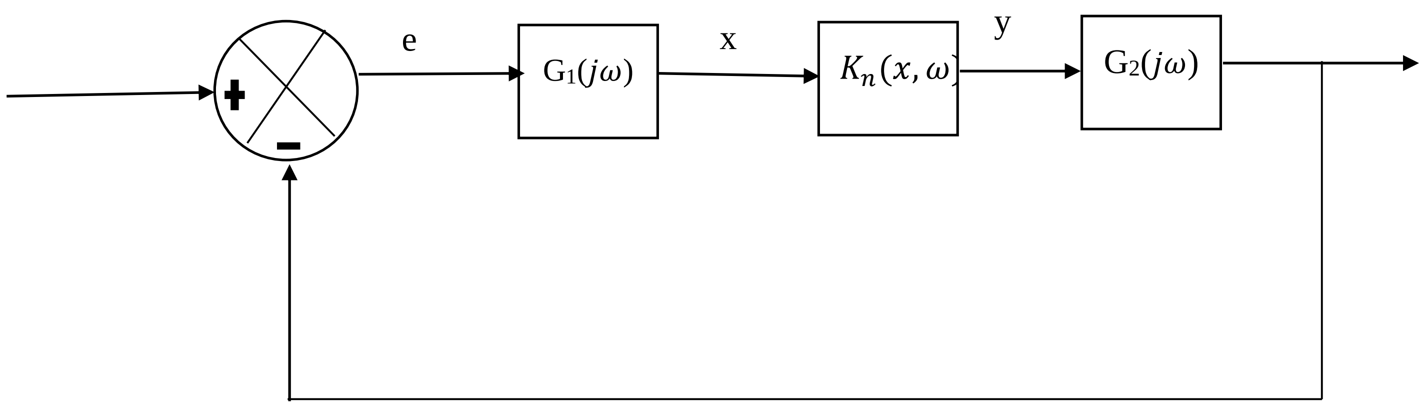
→ This type of linearization is valid for large signals as well so long as the harmonic filtering condition is satisfied.

→ The NL can be replaced by a describing function $K_N(x, \omega)$ which is defined to be the complex function embodying amplification and phase shift of the fundamental frequency component of y relative to x i.e.

$$K_N(x, \omega) = \left(\frac{Y_1}{X}\right) < \phi$$

When the input to the non-linearity is

$$x = xsin\omega t$$



Derivation of Describing Function

The describing function of a non-linear element is given by

$$K_N(x,\omega) = \left(\frac{Y_1}{X}\right) < \phi$$

X- amplitude of the sinusoid

$Y_{1=}$ amplitude of the fundamental harmonic component of the output.

$\phi =$ phase shift of the fundamental harmonic component of the output wrt the input.

→ To compute the describing function of a non-linear element, simply find the fundamental harmonic component of its output for an input $x = xsin\omega t$

The fundamental component of the o/p can be written as

$$y_1 = A_1 sin\omega t + B_1 cos\omega t$$

Where A_1 & B_1 are the coefficients of fourier series.

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} ysin\omega t * d(\omega t)$$

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} ycos\omega t * d(\omega t)$$

The amplitude and phase angle of the fundamental component of the ouput are given by $Y_1 = \sqrt{A_1^2 + B_1^2},\ \phi = \tan^{-1}\left(\frac{B_1}{A_1}\right)$

Pell's method of phase trajectory

The 2nd order non-linear equation is considered to be phase trajectory by using Pell's Method.

General form

$$\frac{d^2x}{dt^2} + \phi\left(\frac{dx}{dt}\right) + f(x) = 0 \qquad (1)$$

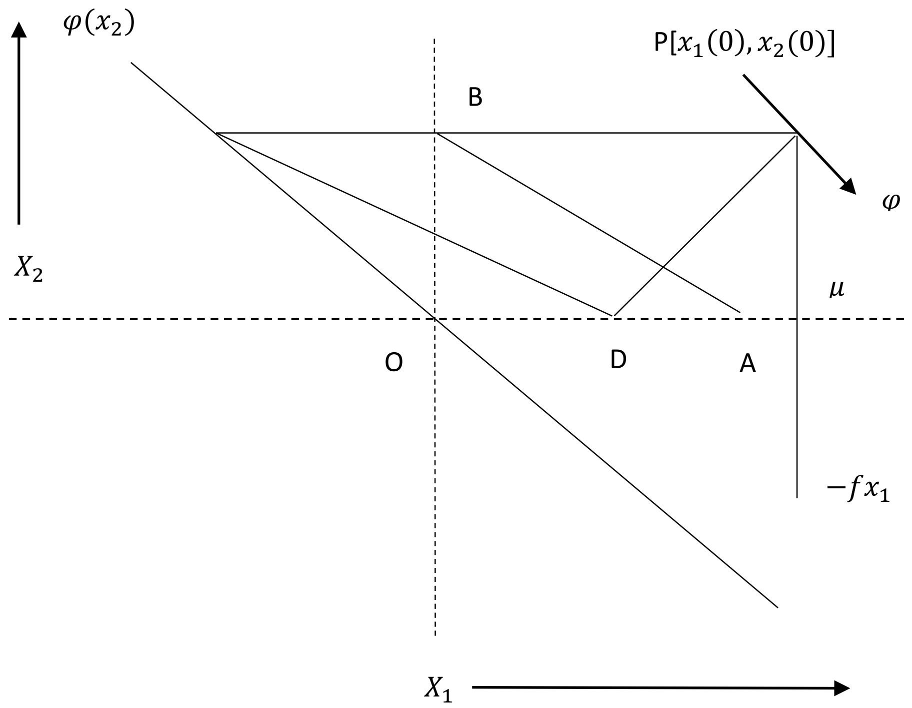
Now defining $x_2 = \frac{dx}{dt} = \frac{dx_1}{dt}$

The above equation can be re-arranged after dividing by x_2 to get

$$\frac{\left(\frac{d^2x}{dt^2}\right)}{x_2} = \frac{x_2}{x_1} = \frac{-\phi(x_2) - f(x_1)}{x_1} \qquad (2)$$

$\rightarrow -\phi(x_2)$ with x_2 and $-f(x_1)$ with x_1

For given initial condition $P[x_1(0), x_2(0)]$, the construction of a segment of the trajectory are shown.



1) Given the initial condition $P[x_1(0), x_2(0)]$ as shown in above figure. Draw a perpendicular from P on the $x_1\ axis$ to get $O\mu = x_1(0)$ and extend the line PM to meet the curve $-fx_1$ at N. such that $\mu N = f\{x_1(0)\}$. Now locate the point A on the x_1 axis such that $\mu A = \mu N$

2) Similarly, for the given $P[x_1(0), x_2(0)]$, draw PB perpendicular on the x_2 *axis* so that OB=μP $x_2(0)$. Now extend PB to meet the $-\varphi x_2$curve at C such that BC = $\varphi\{x_2(0)\}$. Extend the line segment μA on the x_1 axis point D such that AD=BC. Therefore the segment of the x_1 axis is between D and M equals to the absolute values of the function of x_1 and x_2 . that is DM = DA+A μ= $\{x_2(0)\} + f\{x_1(0)\}$
3) Join the points D and P. the slope of the line DP denoted by $m_1 = \frac{PM}{DM} = \frac{x_2}{\varphi(x_2)+f(x_1)}$
4) If the slope of the trajectory in equation (2) is denoted by m_2, then $m_1 m_2 = -1$. The solution of equation (2) will move on a segment of a perpendicular to the line DP to get the next point $\varphi(x_1(f_1), x_2(f_2))$ as shown in the above figure. The process is repeated with φ as the new initial condition to get the complete trajectory.

Advantages of this method

a) The construction is very simple.
b) It eliminates the trial and error approach of the delta method.

Examples of Pell's Method

$$\frac{d^2x}{dt^2} + c_1\frac{dx}{dt} + \omega_1^2(1 + b^2x^2)x = 0$$

Draw the phase plane trajectory using Pell's method

Let $c_1 = 125 \quad \omega_1^2 = 25000 \quad b^2 = 0.60$

We introduce time scaling $\tau = \omega_1 t$ so that

$d\tau = \omega_1 dt \Longrightarrow \omega_1 = \frac{d\tau}{dt}$

$$\Rightarrow dt = \frac{d\tau}{\omega_1}$$

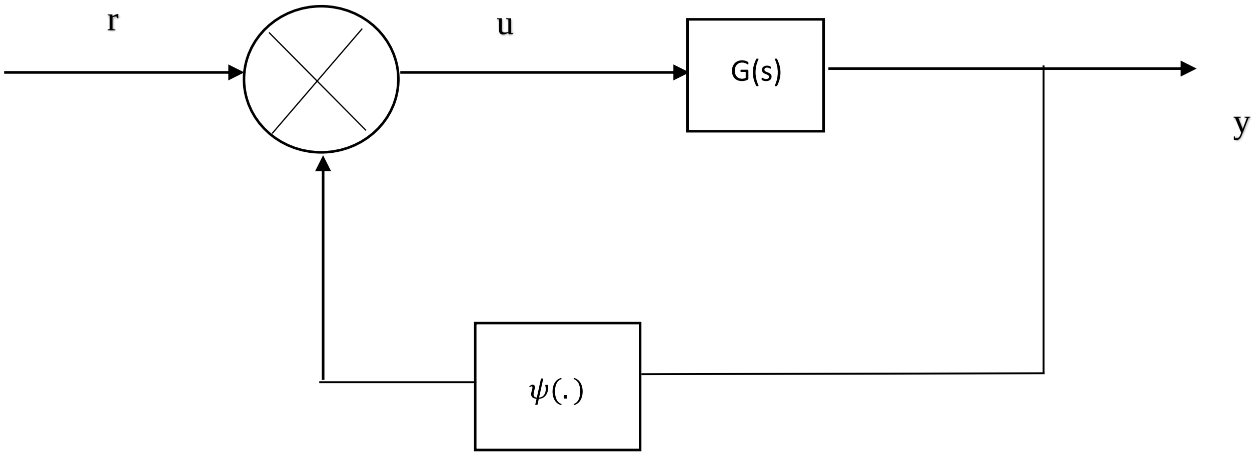
$$\frac{\frac{d^2x}{d\tau^2}}{\omega_1^2} + c_1\frac{\frac{dx}{d\tau}}{\omega_1} + \omega_1^2\left(1 + b^2x^2\right)x = 0$$

$$\omega_1^2\frac{dx^2}{d\tau^2} + c_1\omega_1\frac{dx}{d\tau} + \omega_1^2(1 + b^2x^2)x = 0$$

$$\frac{d^2x}{d\tau^2} + \frac{c_1}{\omega_1}\frac{dx}{d\tau} + (1 + b^2x^2)x = 0$$

MODULE-2

Stability Study of Non-Linear System



→ The process for representing a system is in the form of having a feedback connection of a linear dynamics system and non-linear element.

→ The control system non-linearity is in form of relay or actuator or sensor non-linearity

$$u = -\psi(.) + r$$

→ Here we assume that the external input r is equal to 0 which indicates the behaviour of an unforced system.

→ The system is said to be absolutely stable if it has a globally uniformly asymptotically stable equilibrium points are the origin.

→ For all non-linearities in a given circle, without input, initial condition and asymptotically stable.

→ For this the Circle and Popov criteria give frequency domain absolutely stability in the form of silent positive realness of certain transfer function.

→ In the SISO case both criterion can be applied graphically. We assume the external input $r = 0$ and the behaviour of unforced system can be represented by

$$\dot{X} = Ax + Bu$$

$$Y = CX + Du$$

$$u = \psi(.)$$

Where $X \in R^{\wedge}$ $u, y \in$

$(A, B) - Controllable$

$(A, C) - Observable$

$\psi(0, \infty) \times R^{P}$ is a memory less possibly time varying non-linearity and probability continuous in time t.

Definition-1

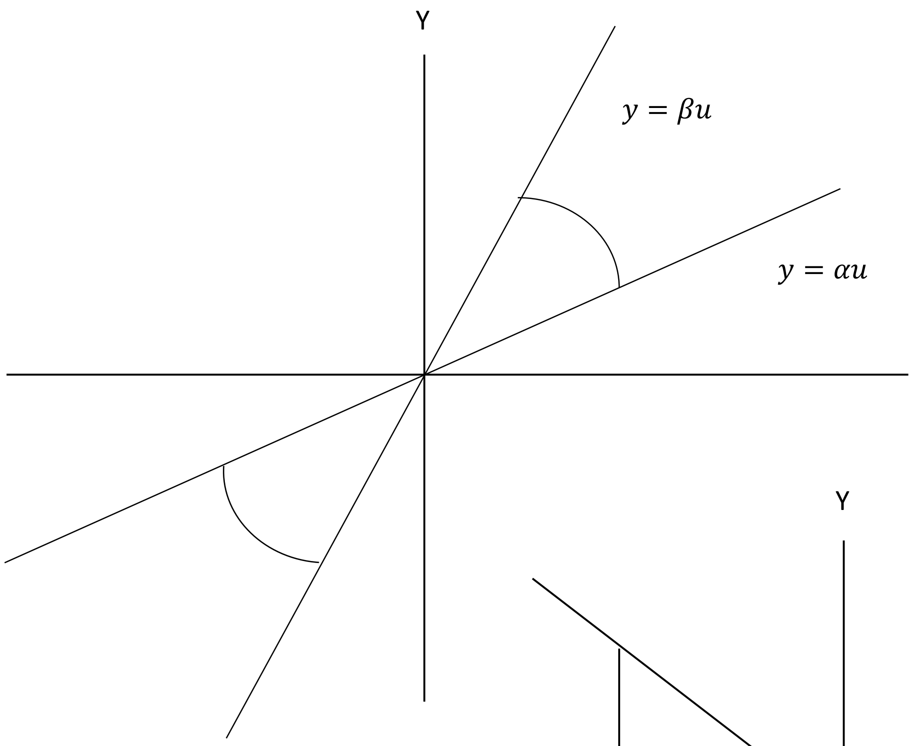
A memory function $h(0, \infty)$

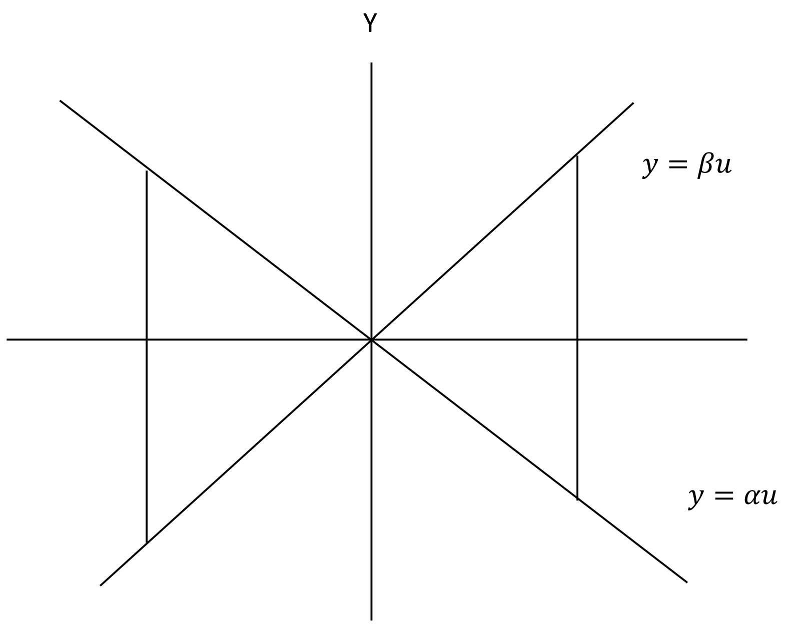
R is said to belong to the sector

$h(0, \infty) \rightarrow$ if $u.h[t, u] \geq 0$

$h(\propto, \infty) \rightarrow$ if $u[h(t, u) - \beta u] \leq 0$

$h(0, \beta) \rightarrow with\ \beta > 0\ if\ [h(t, u) - \beta u] \leq 0$





Definition 2

In closed loop system is called absolute lead stable in sector between $(0, \beta)$ if the origin is globally uniformly asymptotically stable. For any non-linearity in the given sector it is absolutely stable in a finite domain if the origin is unifromly asymptoticall stable.

Popov Criterion (Time Invariant Feedback)

Popov criteria is applicable if the following condition are satisfied.

- Time invariant non-linearity $\psi: R \rightarrow where\ R\ satisfies\ the\ sector\ condition.$
- $\wedge\ where\ R\ satisfies\ \psi(0) = \infty$
- $G(S) = \frac{P(s)}{s^{\wedge}q(n)}$ where degree {P(s)}$< degrees\ \{q(s)\}$
- The poles of G(s) are in LHP or on imaginary axis.
- The system is marginally stable in singular case.

Theorem-1

The closed loop system is absolutely if $\psi \in [0, k]$

$$0 < k < \infty$$

And there exists a constant ‘q’ such that the following condition is satisfied

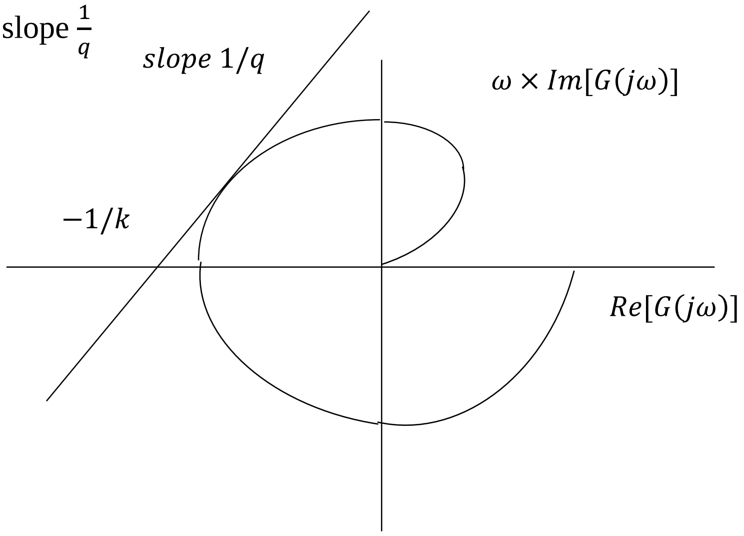
$$Re[G(j\omega)].\, jqn\ Im[G(j\omega)] > -\frac{1}{k}$$

Where $\omega \in [-\infty, \infty]$

Graphical Representation:

Popoc plot $P(j\omega)\ [Z.\, Re[G(j\omega)] + j\omega Im[G(j\omega)]\ \ where\ \omega > 0$

The closed loop system is absolutely stable if "P" lies to the right of the line that interrupts $S\left(-\frac{1}{k} + j0\right)$ will slope $\frac{1}{q}$



As per popov criterion the closed loop system in absolute stable if $\psi \in [0, k], 0 < k < \infty$ and their exists a constant q such that the following equation is satisfied.

$$Re[G(j\omega) - jq\omega Im[G(j\omega)] > -\frac{1}{k}$$

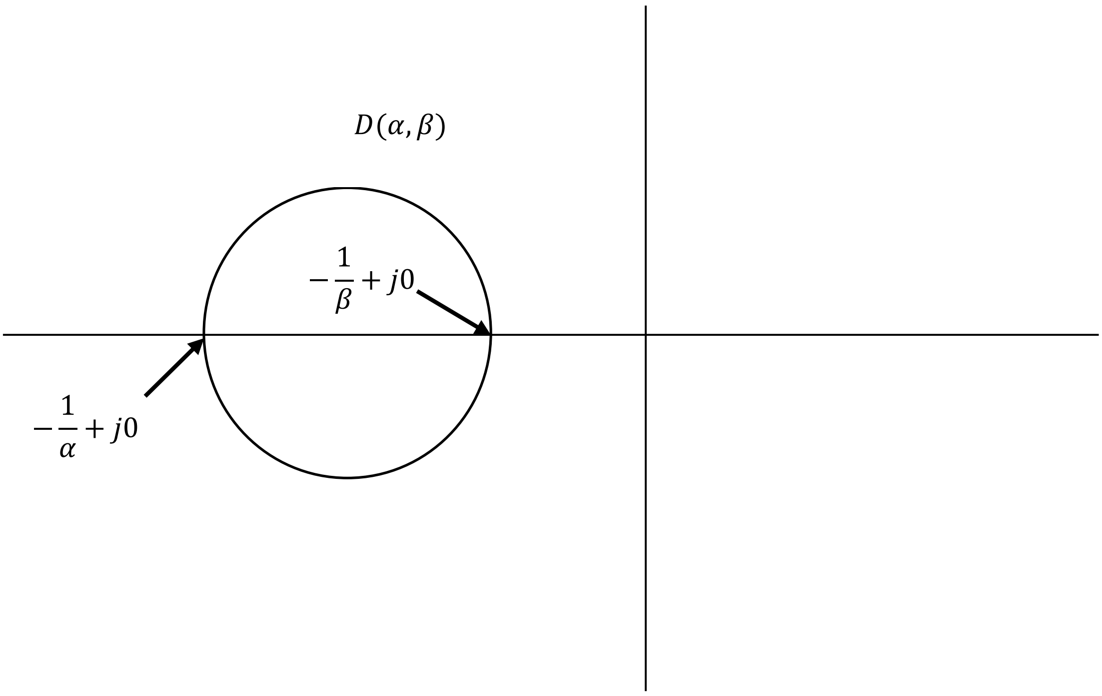
$$where\ \omega\ \in [-\infty, \infty]$$

Circle Criterion:

Applicable to non-linear time variant system which gives information absolutely stability.

Definition

we define $D(\alpha, \beta)$ to be closed disk in the complex plane whose diameter is the line segment connected the points $\left(-\frac{1}{\alpha} + j0\right)$ and $\left(-\frac{1}{\beta} + j0\right)$



Definition 2

Consider a scalar system

$$\dot{X} = Ax + Bu$$

$$Y = Cx + Du$$

$u = -\psi(y, t) \rightarrow$ non-linear time invariant

Where $G(s) = f(A, B, C, D)\ and\ \psi \in [\alpha, \beta]$

The system is absolutely stable if one of the three following conditions :

1) If $o < \alpha < \beta$, the nyquist plot $G(j\omega)$ doesn't enter $D[\alpha, \beta]$ and encircles it in counter clockwise directions where ω is no of poles of $G(s)$ with the real points.
2) If $(\alpha = 0) < \beta, G(s)$ is therefore Hurwitz and Nyquist Plot $G(j\omega)$ lies to right of the vertical line defined by $Re(s) = -\frac{1}{p}$
3) If $\alpha < 0 < \beta$ is Hurwitz and Nyquist plot $G(j\omega)$ the interior part of $D[\alpha, \beta]$

Describing Function

$x = xsin\omega t$

$$y = a_0 + a_1 sin\omega t + b_1 cos\omega t + a_2 sin2\omega t + b_2 cos2\omega t + \cdots + \cdots + a_n \sin(n\omega t) + b_n \cos(n\omega t)$$

$$= a_0 + \sum_{n=1}^{\infty} Y_n \sin(n\omega t + \phi_n)$$

$$= a_0 + Y_1 \sin(\omega t + \phi_1) + Y_2 \sin(2\omega t + \phi_2) + \cdots$$

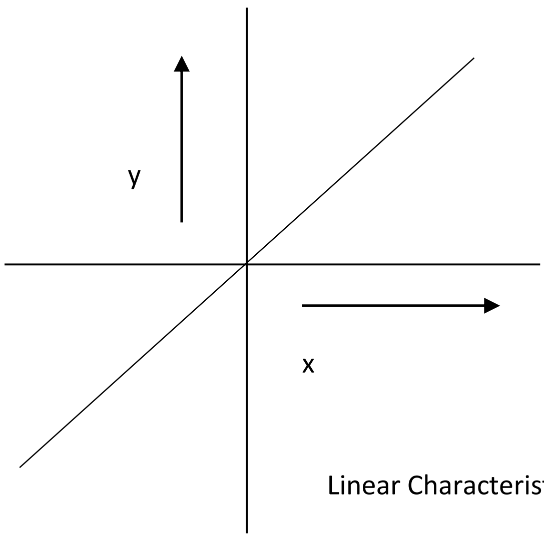
Where $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta = \frac{1}{2\pi}\int_0^{2\pi} y\, d\theta$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta = \frac{1}{\pi}\int_0^{2\pi} ycos(n\theta)\, d\theta$$

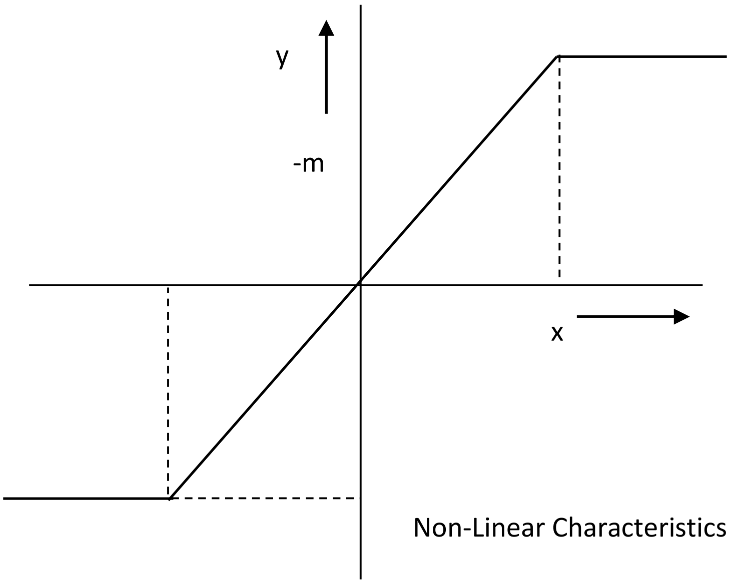
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta) d\theta = \frac{1}{\pi}\int_0^{2\pi} ysin(n\theta)\, d\theta$$

$$Y_n = \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(\frac{a_n}{b_n}\right)$$



Linear Characteristics

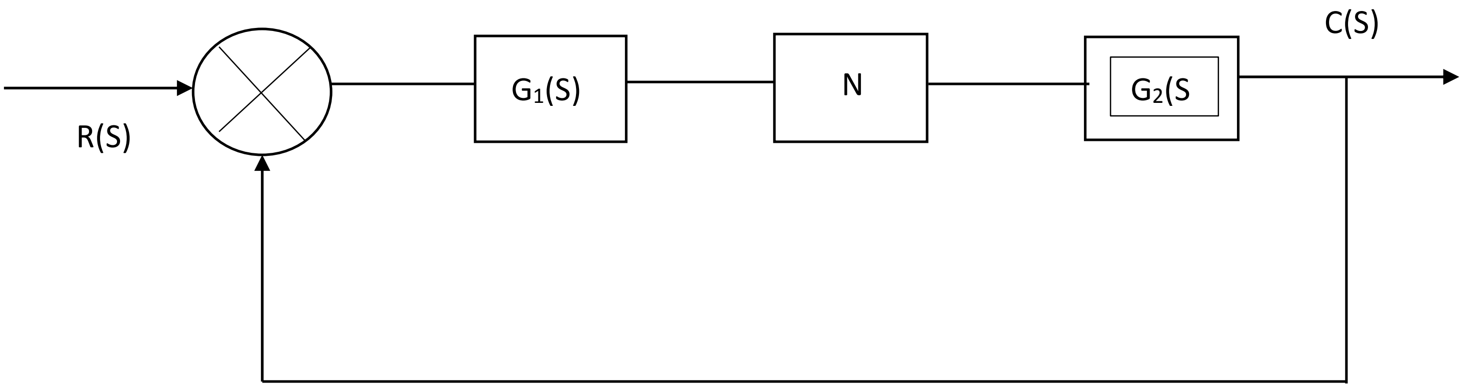


Non-Linear Characteristics

$Y_1 = \sqrt{a_1^2 + b_1^2}$

$\phi_1 = \tan^{-1}\left(\frac{a_1}{b_1}\right)$

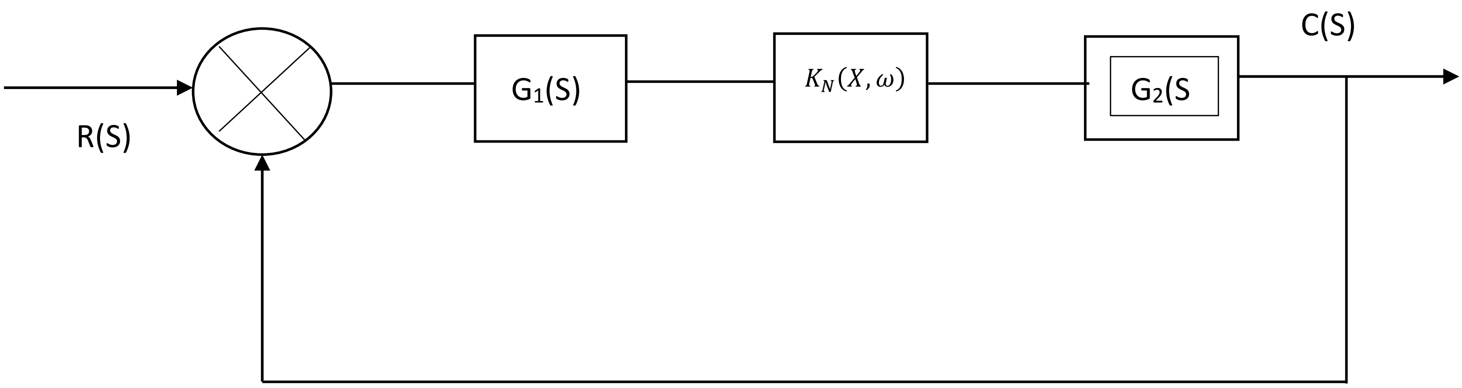
Describing Funtion (DF) = $\frac{Y_1}{X} < \phi_1 = K_N(X, \omega)$

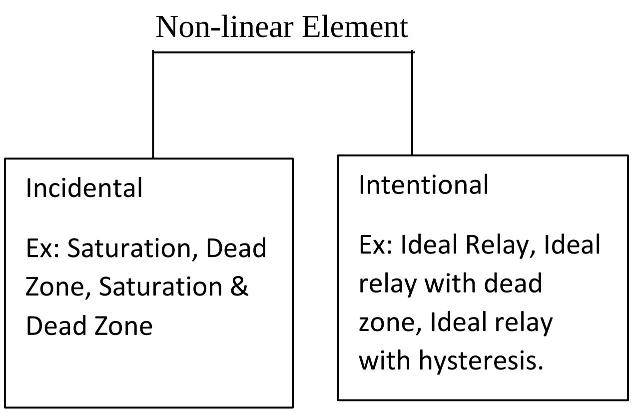


N- Non-Linear Element

N must be linearized by the help of DF represented as $K_N(X, \omega)$

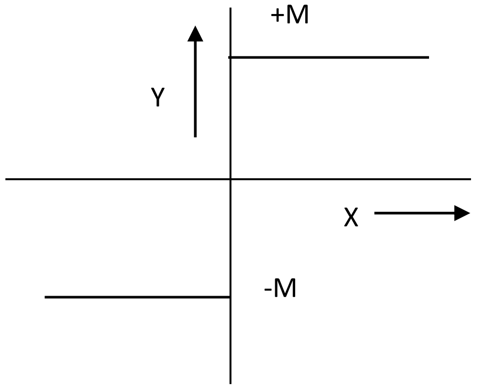
Where $K_N(X, \omega) = \frac{Y_1}{X} < \phi_1$



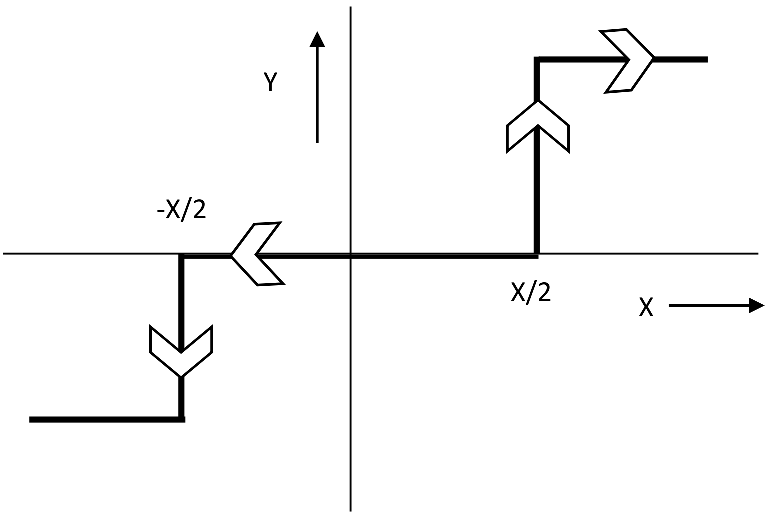


Intentional Non-Linearity

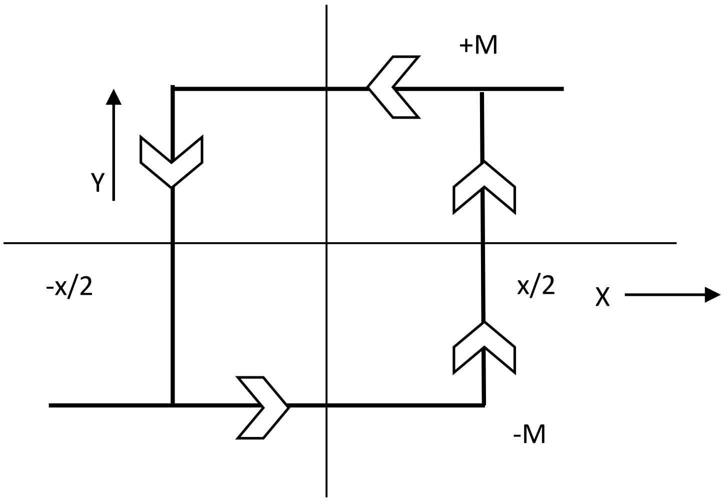
a) Ideal Relay



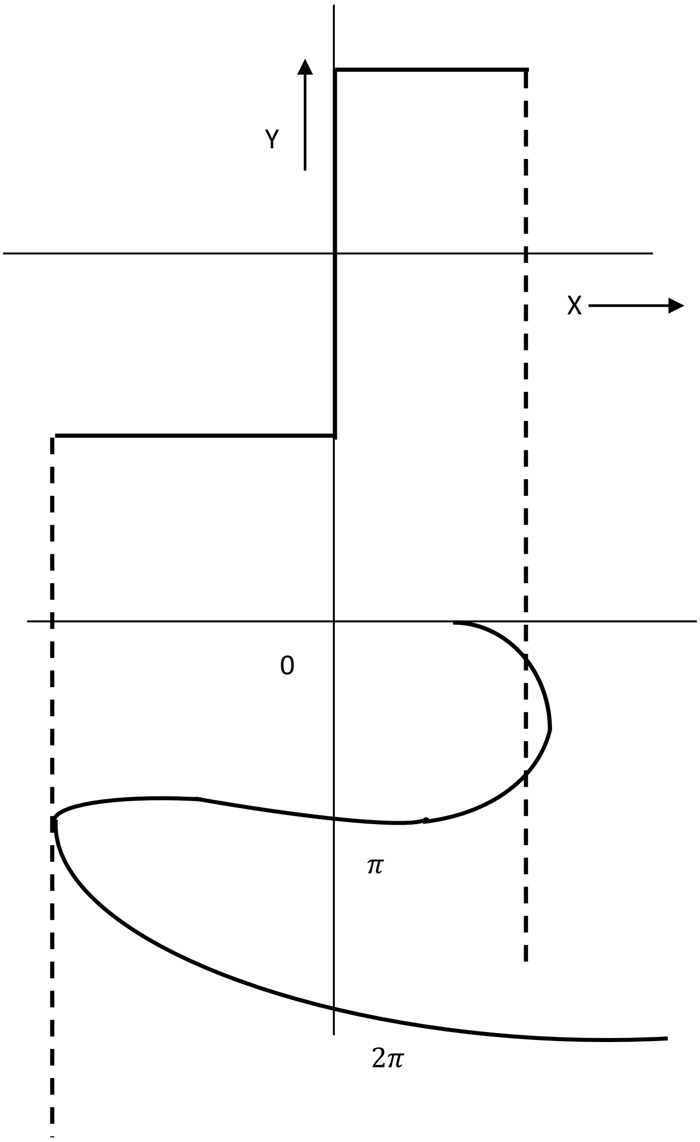
b) Ideal Relay with Deadzone

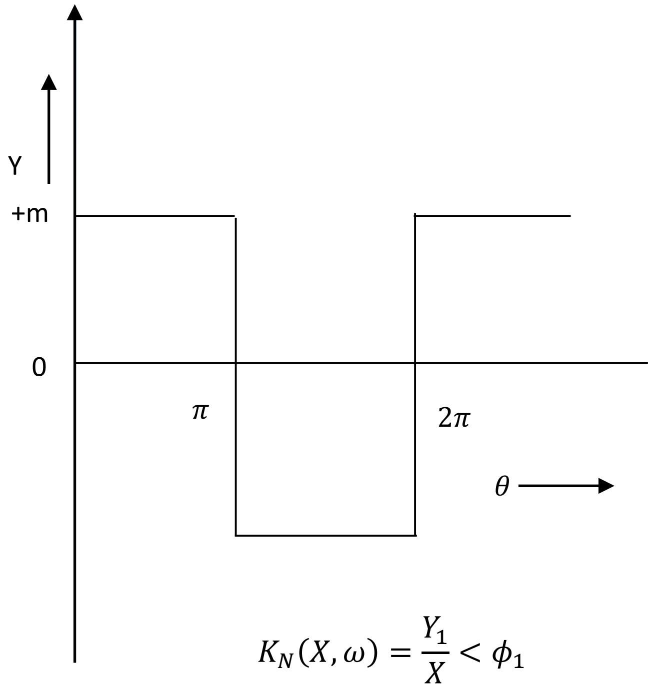


c) Ideal relay with hysteresis



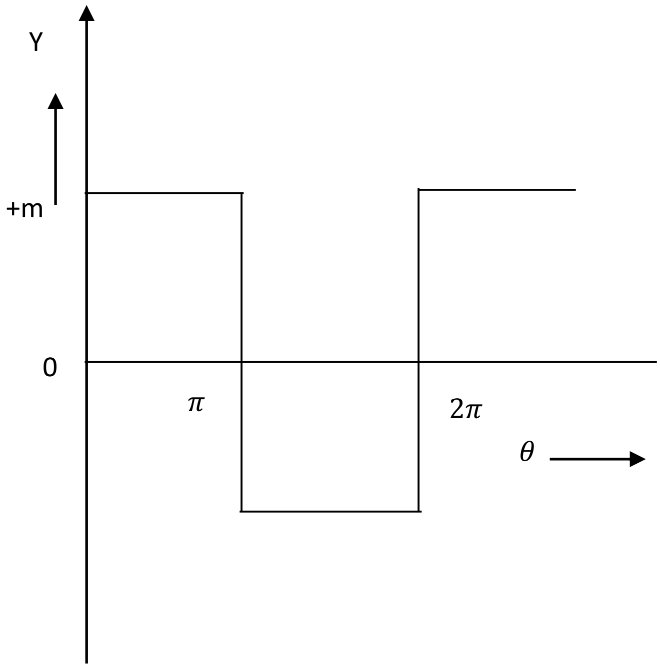
Ideal Relay

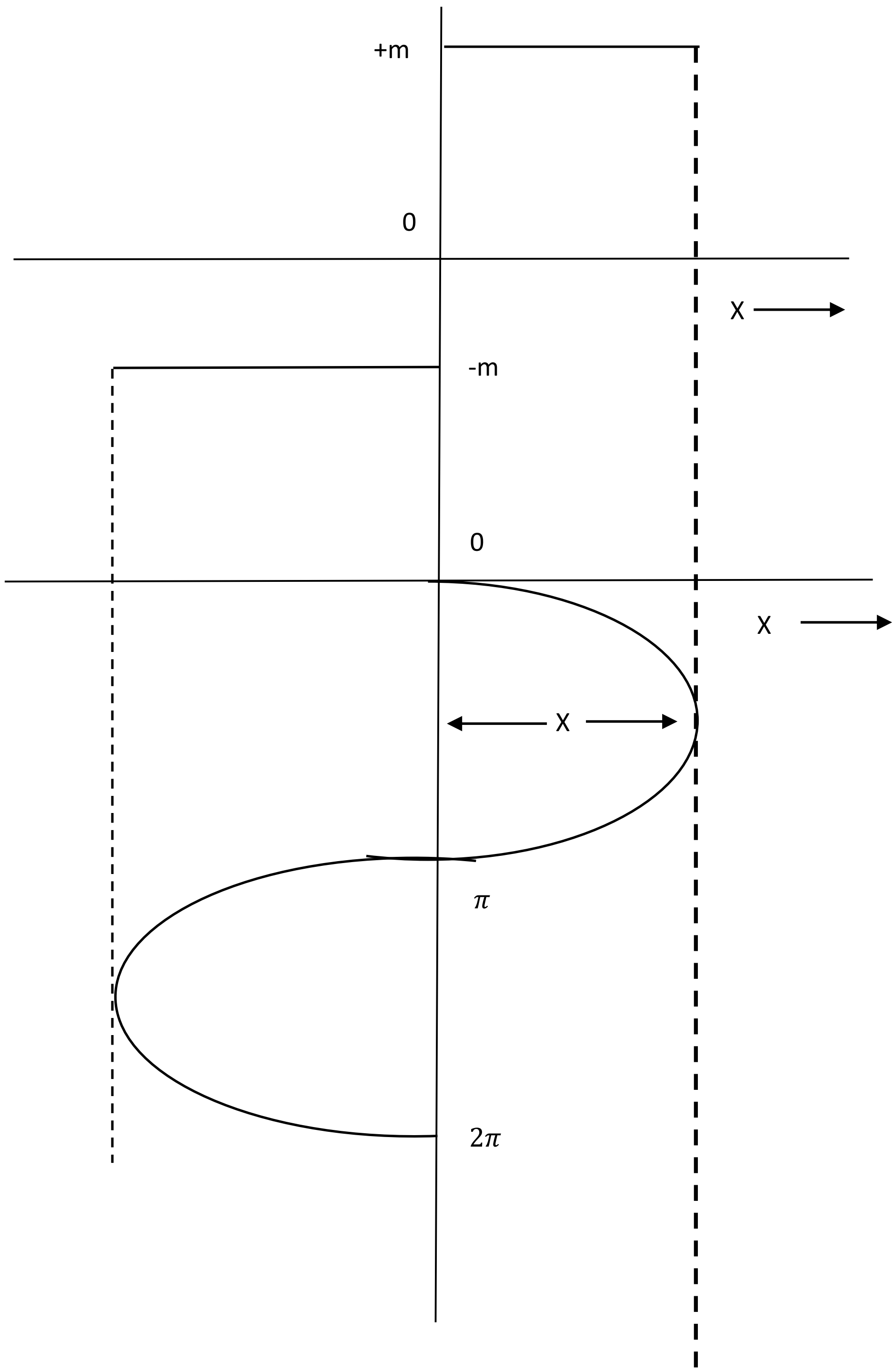




$$K_N(X,\omega) = \frac{Y_1}{X} < \phi_1$$

Ideal Relay (LVDT)





Here $x(t) = xsin\omega t$

$y = M, 0 \leq \theta \leq \pi$

$= -M, \pi \leq \theta \leq 2\pi$

DF= $K_N(x,\omega) = \frac{Y_1}{x} < \phi$ (1)

$Y_1 = \sqrt{a_1^2 + b_1^2}$ & $\phi = \tan^{-1}\left(\frac{a_1}{b_1}\right)$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} Y\, d\theta = \frac{1}{2\pi}\left[\int_0^{\pi} M\, d\theta + \int_0^{2\pi}(-M)\, d\theta\right]$$

$$= \frac{1}{2\pi} M[(\theta)_0^{\pi} - (\theta)_0^{\pi}]$$

$$= \frac{1}{2\pi}.M[\pi - 0 - 2\pi + \pi] = 0$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} Ycos\theta\, d\theta = \frac{1}{\pi}\left[\int_0^{\pi} Mcos\theta\, d\theta + \int_0^{2\pi}(-M)cos\theta\, d\theta\right]$$

$$= \frac{1}{\pi}\ M\left[[sin\theta_0^{\pi}] - [sin\theta_0^{2\pi}]\right]$$

$$= \frac{M}{\pi}[sin\pi - sin0 - sin2\pi + sin\pi] = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} Ysin\theta\, d\theta = \frac{1}{\pi}\left[\int_0^{\pi} Msin\theta\, d\theta + \int_0^{2\pi}(-M)sin\theta\, d\theta\right]$$

$$= \frac{M}{\pi}\left[[-cos\theta_0^{\pi}] - [-cos\theta_0^{2\pi}]\right] = 0$$

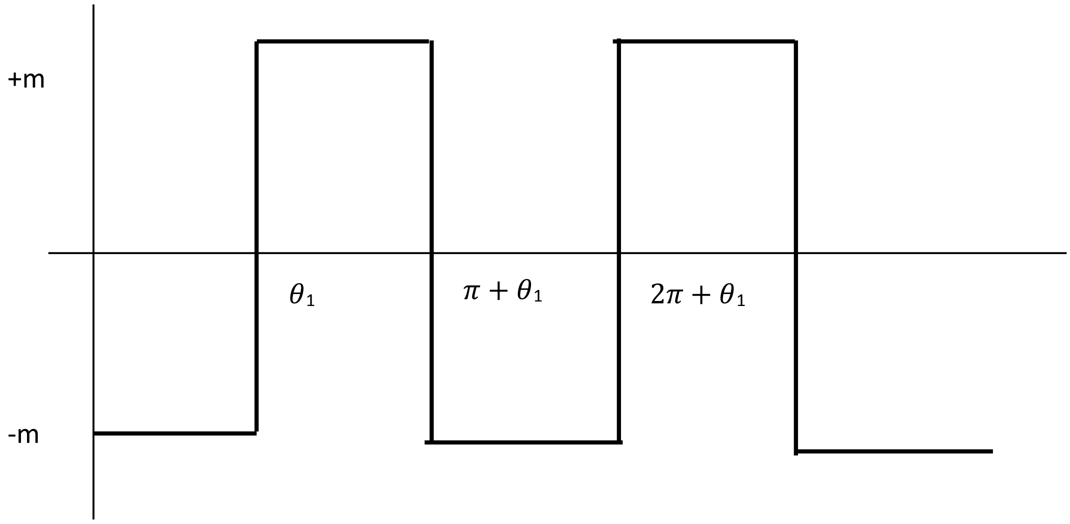
$$= \frac{M}{\pi}[-cos\pi + cos\pi + cos2\pi - cos\pi] = \frac{4M}{\pi}$$

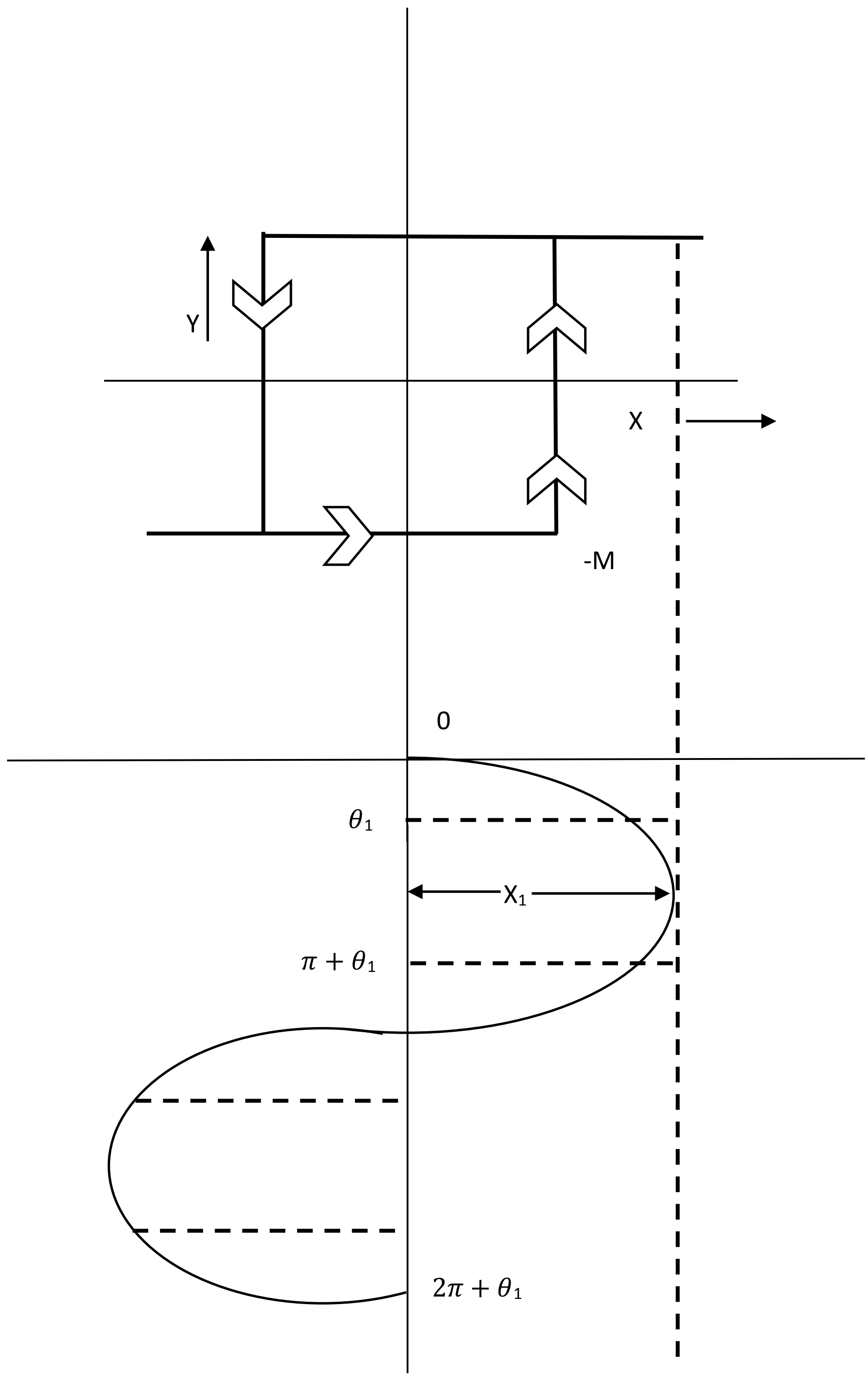
$$\phi = \tan^{-1}\left(\frac{a_1}{b_1}\right) = \tan^{-1}\left(\frac{0}{4m/\pi}\right) = 0$$

$Y_1 = \sqrt{0 + \left(\frac{4M}{\pi}\right)^2} = \frac{4M}{\pi}$

DF = $K_N(x,\omega) = \frac{Y_1}{x} < \phi = \frac{4M}{\pi x} < 0$

Relay with Hysteresis (Schmitt-trigger)





Here $x = xsin\theta$

$$\theta_1 = \sin^{-1}\left(\frac{h}{2\pi}\right)$$

$$Y = M, \theta_1 \leq \theta \leq \pi + \theta_1$$

$$= -M, \pi + \theta_1 \leq \theta \leq 2\pi + \theta_1$$

DF = $\frac{Y_1}{X} < \phi$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} Y\, d\theta = \frac{1}{2\pi}\left[\int_{\theta_1}^{\pi+\theta_1} M\, d\theta - \int_{\pi+\theta_1}^{2\pi+\theta_1} M\, d\theta\right]$$

$$= \frac{M}{2\pi}\left[\theta \,\Big|_{\theta_1}^{\pi+\theta_1} - (\theta)\Big|_{\pi+\theta_1}^{2\pi+\theta_1}\right]$$

$$= \frac{M}{2\pi}[\pi + \theta_1 - \theta_1 - 2\pi - \theta_1 + \pi + \theta_1] = 0$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} Ycos\theta\, d\theta = \frac{1}{\pi}\left[\int_{\theta_1}^{\pi+\theta_1} Mcos\theta\, d\theta + \int_{\pi+\theta_1}^{2\pi+\theta_1} (-Mcos\theta\, d\theta)\right]$$

$$= \frac{M}{\pi}\left[\int_{\theta_1}^{\pi+\theta_1} cos\theta\, d\theta - \int_{\pi+\theta_1}^{2\pi+\theta_1} (cos\theta\, d\theta)\right]$$

$$= \frac{M}{\pi}\left[sin\theta \,\Big|_{\theta_1}^{\pi+\theta_1} - (sin\theta)\Big|_{\pi+\theta_1}^{2\pi+\theta_1}\right]$$

$$= \frac{M}{\pi}[\sin(\pi + \theta_1) - sin\theta_1 - \sin(2\pi - \theta_1) + \sin(\pi + \theta_1)] = 0$$

$$= \frac{M}{\pi}[-\sin\theta_1 - sin\theta_1 - \sin\theta_1 - \sin\theta_1)] = \frac{4M\sin\theta_1}{\pi} = -\frac{4M}{\pi}\left(\frac{h}{2x}\right)$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} Ycos\theta\, d\theta$$

$$= \frac{1}{\pi}\left[\int_{\theta_1}^{\pi+\theta_1} Msin\theta\, d\theta + \int_{\pi+\theta_1}^{2\pi+\theta_1} (-Msin\theta\, d\theta)\right]$$

$$= \frac{M}{\pi}\left[(-cos\theta) \,\Big|_{\theta_1}^{\pi+\theta_1} + cos\theta\Big|_{\pi+\theta_1}^{2\pi+\theta_1}\right]$$

$$= \frac{M}{\pi}[-\cos(\pi + \theta_1) + cos\theta_1 + cos(2\pi + \theta_1) - \cos(\pi + \theta_1)] = 0$$

$$= \frac{M}{\pi}[\cos\theta_1 + cos\theta_1 + \cos\theta_1 + \cos\theta_1)] = \frac{4M\cos\theta_1}{\pi} = \frac{4M}{\pi}\left[1 - \left(\frac{h}{2x}\right)^2\right]^{1/2}$$

$$Y_1 = \sqrt{a_1^2 + b_1^2} = \sqrt{\left(\frac{4M}{\pi}\,\frac{h}{2x}\right)^2 - \left(\frac{4M}{\pi}\left(1 - \frac{h}{2x}\right)^2\right)^{1/2}}$$

$$= \frac{4M}{\pi}\sqrt{\left(\frac{h}{2x}\right)^2 + 1 - \left(\frac{h}{2x}\right)^2} = \frac{4M}{\pi}$$

$$\phi = \tan^{-1}\left(\frac{a_1}{b_1}\right) = \tan^{-1}\left[\frac{\frac{4M}{\pi}sin\theta_1}{\frac{4M}{\pi}\cos\theta_1}\right] = \tan^{-1}\tan\theta_1$$

$$\phi = \sin^{-1}\left(\frac{h}{2x}\right) \; and \; DF = K_N(x, \omega) = \frac{4M}{\pi} < \sin^{-1}\left(\frac{h}{2x}\right)$$

MODULE-III

Liapunov Function for Non-Linear System

Krasovskii's Method:

Consider the system

$$\dot{x} = f(x); f(0) = 0 \qquad (1) \qquad [singular\ point\ at\ origin]$$

Define a Liapunov Function as $v = f^T P f$ (2)

Where P= Symmetric +ve definite matrix

Now $\dot{v} = \dot{f}^t p f + f^t p f$ (3)

$$\left(\dot{f} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t}\right) = Jf$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \frac{\partial f_n}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \frac{\partial f_2}{\partial x_n} \\ \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \ldots\ldots\ldots\ldots\ldots\ldots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Replace $\dot{f}$ in equation (3) we have

$$V = (Jf)^T P f + f^t P J f$$

$$= f^T J^T P F + f^t P J f = f^T (J^T P + PJ) f$$

$$Q = J^T P + PJ$$

Since V is +ve definite for the system to be asymptotically stable, Q should be positive definite. If V(x)→ ∞ as $\|x\| \to \infty$ then the system is asymptotically stable in large.

Variable Gradient Method

For the autonomous system

$$\dot{x} = f(x); f(0) = 0 \qquad (1)$$

Let V(x) be considered for a LF

The time derivative of V can be expressed as

$$\dot{V}(x) = \frac{\partial v}{\partial x_1} \dot{x}_1 + \frac{\partial v}{\partial x_2} \dot{x}_2 + \cdots \ldots\ldots\ldots\ldots\ldots\ldots. + \frac{\partial v}{\partial x_n} \dot{x}_n \qquad (2)$$

Which an be expressed in terms of the gradient of V

As $V = (\nabla V) T_{\dot{X}}$ (3)

$$\nabla V = \begin{bmatrix} \frac{\partial v}{\partial x_1} = \nabla V_1 \\ \frac{\partial v}{\partial x_2} = \nabla V_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \frac{\partial v}{\partial x_n} = \nabla V_n \end{bmatrix}$$

The LF can be generated by integrating w.r.t. time both sides equation (3)

$$V = \int_0^x \frac{\partial v}{\partial t}\, dt = \int_0^x (\nabla v)^T dx \qquad (4)$$

→ The above integral is a line integral whose result is independent of the path.

→ The integral can be evaluated sequentially along the component directions (x_1, x_2, ……..x_n) of the state vector.

That is $V = \int_0^x (\nabla v)^T dx$

$$V = \int_0^{x_1} \nabla V_1(x_1, 0,0,0 \dots \dots .0) dx_1 + \int_0^{x_2} \nabla V_2(x_1, x_2, 0,0,0 \dots \dots .0) dx_2$$
$$+ \int_0^{x_3} \nabla V_3(x_1, x_2, x_3, 0,0,0 \dots \dots .0) dx_3 + \cdots \dots \dots \dots$$
$$+ \int_0^{x_n} \nabla V_n(x_1, x_2, \dots \dots . x_n) dx_n \qquad (5)$$

Let us define

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \dots \dots \dots \dots \dots \dots \dots . e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The integral given in equation (5) states that the path starts from the origin and moves along the vector e_1 to x_1. From this point the path moves in the direction of the vector e_2 to x_2. In this way the path finally reaches the point $(x_1, x_2, \dots \dots . x_n)$

* From the function V to be unique thecurl of its gradient must be zero

$$\nabla \times (\nabla V) = 0 \qquad (6)$$

$\rightarrow$ This results in $\frac{n}{2}(n-1)$ equation to be satisfies by the components of the gradient

$$\left(\frac{\partial \nabla V_i}{\partial x_j}=\frac{\partial \nabla V_j}{\partial x_i}\right) \text{ for all } i,j \qquad (7)$$

To begin with a completely general form given below is assumed for the gradient vector ∇V

$$\nabla V=\begin{bmatrix}\nabla V_1\\ \nabla V_2\\ \vdots\\ \vdots\\ \nabla V_n\end{bmatrix}=\begin{bmatrix}a_{11}x_1+a_{12}x_2+\cdots\ldots\ldots\ldots\ldots a_{1n}x_n\\ a_{21}x_1+a_{22}x_2+\cdots\ldots\ldots\ldots\ldots a_{2n}x_n\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ a_{n1}x_1+a_{n2}x_2+\cdots\ldots\ldots\ldots\ldots a_{nn}x_n\\ \end{bmatrix}$$

$a_{ij}'s$ *are completely undetermined* quantity and could be constant of fucntions of both state variable and t. it is convenient to choose as a constant

Advantages:

Easy to apply

Disadvantages:

- Only asymptotic stability can be investigated.
- F(x); continuously differentiable
- Domain of attraction is unknown

The determination of Liapunov stability via Liapunov direct method centres around the choice of positive definite function v(x) called the Liapunov function.

There is no universal method for selecting Liapunov function whch is unique for a special problem.

Theorem-1

$$\dot{x}=f(x);f(0)=0$$

Real scalar function v(x) satisfies the following properties for all x on the region $\|x\|<t$

a) $V(x)>0;\ \ x\neq o$

b) $V(0)>0$

c) $V(x)$ has continuous partial derivatives with respect to all components x.

d) $\frac{dv}{dt} < 0 \left(i.e \frac{dv}{dt} \;\; is - ve\ semi\ definite\ scalar\ function\right)$

Then the system is stable.

Theorem-2

a) $V(x) > 0;$

b) $V(0) > 0$

c) $V(x)$ has continuous partial derivatives with respect to all components x.

d) $\frac{dv}{dt} < 0 \left(i.e \frac{dv}{dt} \;\; is - ve\ semi\ definite\ scalar\ function\right)$

$$\dot{X} = f(x);\ \ f(0) = 0$$

There exists a real scalar function $w(x)$

1) $w(x) > 0\ ; x \neq 0$
2) w(0)= 0
3) $w(x)$ has continuous partial derivatives with respect to all components x.
4) $\frac{dw}{dt} \geq 0$

Then the system is unstable at the origin

Analysis of Stability by Liapunov Direct Method

→ This method is useful for determining the stability. It is also applicable for linear systems.

→ In this method it can be determined without actually solving the differential equation. So it is called as direct method.

Concept of Definiteness

Let V(x) is a real scalar function. The scalar function v(s) is said to be positive definite if the function V(x) has always +ve sign in the given region about the origin except only at origin where it is zero

- The scalar function V(x) is the +ve in the given region if V(x)>0 for all non-zero states x in the region and v(0) = 0

- The scalar function V(x) is said to be negative definite if the function V(x) has always –ve sign in the given region about the origin, except only at the origin where it is zero or a scalar function.

Liapunov stability Theorem

$$\dot{X} = f(x);$$

If there exist a scalar function V(x) which is real, continuous and has continuous first partial derivatives with

a) $V(x) > 0;\ \ x \neq 0$
b) $V(0) > 0$
c) $V(x) < 0$ for all $x \neq 0$

Then the system is asymptotically stable. V(x) is the Liapunov Function.

Ex: The system is given by

$\dot{x}_1 = x_2$

$x_2 = -x_1 - x_2^3$

Investigate the system by Liapunov method using

$V = x_1^2 + x_2^2$

Solution:

$$\dot{V}(x) = \frac{\partial v}{\partial x_1}\,\dot{x}_1 + \frac{\partial v}{\partial x_2}\,\dot{x}_2$$

$\dot{V}(x) = 2x_1\,\dot{x}_1 + 2x_2\,\dot{x}_2 = 2x_1x_2 + 2x_2(-x_1 - x_2^3)$

$= -2x_2^4$(-ve)

$\dot{V}(x) < 0 \quad for\ all\ non-zeroes\ i.e.-ve\ definite$

So asymptotically stable.

* A system is describes by
  $\dot{x} = f(x)$

If there exists a scalar function V(x) which is real, continuous and has continuous first partial derivatives with

a) $V(x) < 0;\ \ x \neq 0$
b) $V(0) = 0$
c) $V(x) > 0$ for all $x \neq 0$

Then system is unstable.

* **The direct method of Liapunov and Linear System:**

Let the system is given by

$$\dot{x} = Ax \qquad (1)$$

Select the Liapunov function as

$V(x) = x^T P x$

$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$

Substitute the value of $\dot{x}$ from equation (1)

$\dot{V}(x) = (Ax)^T P x + x^T P A x$

$= x^T A^T P x + x^T P A x$

$= x^T [A^T P + PA] x$

$\Rightarrow \dot{V}(x) = x^T Q x$

where $Q = A^T P + PA \qquad (2)$

Q is +ve definite matrix

Select a +ve definite Q and find out P from equation (2). If P is +ve definite then the system will be stable

Procedure:

Step-1 Select Q as positive definite.

Step-2 Obtain P from the equation (2) from this we will have to solve $\frac{n(n+1)}{2}$ number of equation where n is the order of matrix A.

Step-3 Using Sylvester's theorem determine the definiteness of P. if P is +ve definite the system is stable otherwise unstable. Generally, Q can be taken as Identity Matrix.

Sylvester's Theorem:

This theorem states that the necessary and sufficient condition that the quadratic form V(x) be +ve definite are that all the successive principal minor of P be +ve i.e

$$P_{11} > 0$$

$$\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} > 0 \qquad \begin{bmatrix} P_{11} \dots\dots\dots\dots\dots & \\ P_{1N} \dots\dots\dots\dots & \end{bmatrix} > 0$$

A) V(x) is –ve definite if –V(x) is +ve definite.
B) V(x) = $x^T P x$ is +ve semi definite if P is singular and all the principal minors are non-negative.

Ex: Determine the stability of the system describe by the following equation

$$\dot{x} = Ax \qquad A = \begin{bmatrix} -1 & -2 \\ -1 & -4 \end{bmatrix}$$

$$A^T P + PA = -Q \qquad (Q = I)$$

$$\begin{bmatrix} -1 & 1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix} = -1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

We have taken $P_{12} = P_{21}$

Because solution matrix P is known to be +ve definite real symmetric for stable system.

$$\begin{bmatrix} -P_{11} + P_{21} & -P_{12} + P_{22} \\ -2P_{11} - 4P_{21} & 2P_{12} - 4P_{22} \end{bmatrix} + \begin{bmatrix} -P_{11} + P_{12} & -2P_{11} + 4P_{12} \\ P_{21} + P_{22} & -2P_{21} + 4P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-2P_{11} + 2P_{12} = -1$$

$$-4P_{21} - 8P_{22} = -1$$

$$-2P_{11} - 5P_{12} + P_{22} = 0$$

Solving for $P, we\ get$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 23/60 & -7/60 \\ -\frac{7}{60} & \frac{11}{60} \end{bmatrix}$$

The necessary and sufficient conditions for a matrix

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & \dots\dots\dots\dots\dots\dots\dots . q_{1n} \\ q_{21} & q_{22} & q_{23} & \dots\dots\dots\dots\dots\dots\dots\dots q_{2n} \\ & & . & \\ & & . & \\ & & . & \\ & & . & \\ & & . & \\ & & . & \\ q_{21} & q_{22} & q_{23} & \dots\dots\dots\dots\dots\dots\dots\dots q_{2n} \end{bmatrix}$$

To be +ve definite are that all the successive principal minors of Q be +ve i.e

$q_{11} > 0,\quad \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} > 0,\quad \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{1} \end{bmatrix} > 0$

Def$[Q] > 0$

* The matrix Q is semi definite if any of the above determinants is zero.
* The matrix Q is negative definite (semi-definite) if the matrix –Q is positive definite (Semi-definite)
* If Q is positive definite so Q^2 *and* Q^{-1}. It should be noted that the definiteness of quadratic form scalar function is global.